1 Solution: Planck's units (10 points)

(a) From dimensional analysis

$$c \equiv \left[L^1 M^0 T^{-1}\right] \quad \mathbf{1}$$
$$G \equiv \left[L^3 M^{-1} T^{-2}\right] \quad \mathbf{1}$$
$$\hbar \equiv \left[L^2 M^1 T^{-1}\right] \quad \mathbf{1}$$
$$k_B \equiv \left[L^2 M^1 T^{-2} \Theta^{-1}\right] \quad \mathbf{1}$$
$$G\hbar \equiv \left[L^5 M^0 T^{-3}\right]$$
$$\frac{G\hbar}{c^3} \equiv \left[L^2 M^0 T^0\right]$$
$$\therefore L_p = \sqrt{\frac{G\hbar}{c^3}} \quad \mathbf{1.5}$$
$$\frac{\hbar c}{G} \equiv \left[L^0 M^2 T^0\right]$$
$$\therefore m_p = \sqrt{\frac{\hbar c}{G}} \quad \mathbf{1.5}$$
$$\frac{L_p}{c} \equiv \left[L^0 M^0 T^1\right]$$
$$\therefore t_p = \sqrt{\frac{\hbar G}{c^5}} \quad \mathbf{1.0}$$
$$T_p = \frac{L_p^2 m_p}{k_B t_p^2} = \frac{G\hbar}{k_B c^3} \times \frac{c^5}{\hbar G} \times \sqrt{\frac{\hbar c}{G}}$$
$$\therefore T_p = \sqrt{\frac{\hbar c^5}{G k_B^2}} \quad \mathbf{1.0}$$
$$4\pi\epsilon_0 \equiv \left[L^{-3} M^{-1} T^2 Q^2\right]$$
$$\hbar c \equiv \left[L^3 M^1 T^{-2}\right]$$
$$\therefore q_p = \sqrt{4\pi\epsilon_0 \hbar c} \quad \mathbf{1.0}$$

2 Solution: Circumbinary planet (10 points)

Equating gravitational force to centrifugal force, we get:

$$\frac{4\pi^2}{T_b^2} a_b = \frac{G 2 M_\odot}{4 a_b^2}$$
$$\frac{T_b^2}{a_b^3} = 2\frac{4\pi^2}{G M_\odot}$$
$$\Longrightarrow a_b = 2\,\text{au}. \quad \mathbf{2.0}$$

The period of planet can be calculated from the 3rd Kepler's law of motion

$$\frac{T_p^2}{a_p^3} = \frac{4\pi^2}{G M_\Sigma} = \frac{4\pi^2}{4 G M_\odot}$$
$$\Longrightarrow T_p = 44.7\,\text{years} \quad \mathbf{2.0}$$
$$\text{Now, } \frac{1}{T_s} = \frac{1}{T_b} - \frac{1}{T_p}$$
$$\therefore T_s = \frac{T_b T_p}{T_p - T_b} = 4.4\,\text{years} \quad \mathbf{2.0}$$

One star always covers exactly the half of the planet's surface. In the best case, the stars are located at the maximal angular distance from each other. So the fraction of illuminated surface is simply given by

$$n = \frac{1}{2} + \frac{\theta}{2\pi} = \frac{1}{2} + \frac{2arctan\left(\frac{a_b}{a_p}\right)}{2\pi} = 0.53. \quad \textbf{4.0}$$

3 Solution: Expanding ring nebula (10 points)

(a) The distance traveled by the gas particles of inner and outer radius can be calculated as follows:

$$S_{out} = d_{\oplus}(\theta'_{\text{out}} - \theta_{\text{out}}) = 100pc \cdot (8.0' - 4.0') = 24 \times 10^3\,\text{au}$$
$$S_{in} = d_{\oplus}(\theta'_{\text{in}} - \theta_{\text{in}}) = 100pc \cdot (7.0' - 3.5') = 21 \times 10^3\,\text{au} \quad \textbf{2.0}$$
$$\therefore v_{max} = \frac{S_{out}}{t_0} = 57.0\,\text{km/s}$$
$$v_{min} = \frac{S_{in}}{t_0} = 50.0\,\text{km/s} \quad \textbf{2.0}$$

(b) At the inner edge, escape velocity from the white dwarf can be maximal ($1.44M_{\odot}$- Chandrasekhar limit)

$$v_{esc} = \sqrt{\frac{2G \cdot 1.44M_{\odot}}{S_{in}}} \approx 0.35\,\text{km/s} \quad \textbf{2.0}$$

Since $v_{esc} \ll v_{min}$, the no-gravity scenario can be applied. YES **1.0**
(c)

$$\phi = 8' - 7.5' = 0.5' = 30'' \geq \frac{1.22\lambda}{D}$$
$$\therefore D \geq \frac{1.22\lambda}{\phi} = \frac{1.22 \times 5 \times 10^{-7} \times 206265}{30} = 4.19\,\text{mm} \quad \textbf{2.0}$$

Hence YES **1.0**

4 Solution: Journey Between Galaxies (10 points)

(a) By the Hubble's Law, the rate of recession is

$$v(t) = \frac{\Delta d}{\Delta t} = Hd(t) \quad \textbf{1.0}$$

Thus, by referring to the given relation,

$$d(t) = Ce^{Ht}$$

at time $t = 0$, $d(0) = d_0 = C$. Thus, the distance at a time t:

$$d(t) = d_0 e^{Ht} \quad \textbf{1.0}$$

(b) It is easier to solve this problem, if we consider coordinate system that changes its scale in such a way that distance between earth and our destination does not change (called co-moving frame). Let $v(t)$ be the velocity in this frame at time t and let $l(t)$ be the total distance traveled until the moment t.

$$v(t)d(t) = v_0 d_0$$
$$v(t) = v_0 \frac{d_0}{d(t)} = v_0 \frac{d_0}{d_0 e^{Ht}}$$
$$v(t) = \frac{\Delta l}{\Delta t} = -v_0 e^{-Ht} \quad \textbf{3.0}$$

$$l(t) = -\frac{v_0}{-H} e^{-Ht} + C$$

for specific cases:

$$l(t_0) = \frac{v_0}{H} e^{-Ht_0} + C$$

$$l(0) = \frac{v_0}{H} + C$$

subtracting yields:

$$l(0) = \frac{v_0}{H} - \frac{v_0}{H}e^{-Ht_0}$$

thus:

$$\therefore l_{\text{Total}} = d_0 = -\frac{v_0}{H}(e^{-Ht_0} - 1) \qquad \mathbf{2.0}$$

Rearranging,

$$t_0 = -\frac{1}{H}ln\left(1 - \frac{Hd_0}{v_0}\right) \qquad \mathbf{1.0}$$

Thus, condition for reaching the planet at all:

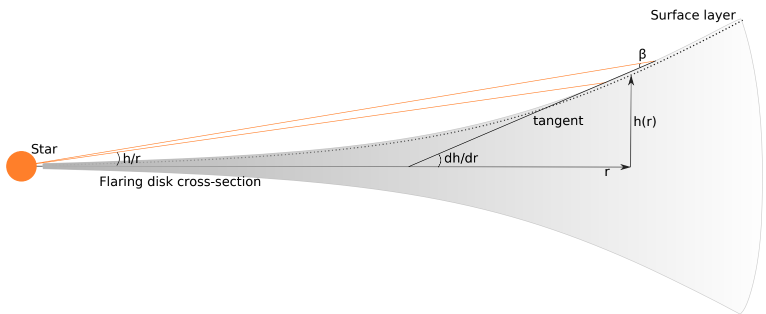
$$1 - \frac{Hd_0}{v_0} > 0$$

$$v_0 > Hd_0 = 70\,\text{km/s} \qquad \mathbf{1.0}$$

(However at this speed it will take eternity to reach it)
For $v_0 = 1000\,\text{km/s}$ we can reach the planet, in:

$$t_0 \approx 5.3 \times 10^{16}\,\text{s} = 17\,\text{Gyr} \qquad \mathbf{1.0}$$

5 Solution: Flaring protoplanetary disk (10 points)



(a) The angle between the light beam and the horizontal plane is

$$\theta_l \approx \tan\theta_l = \frac{h(r)}{r} \qquad \mathbf{1.0}$$

Additionally, a tangent is inclined to the disc plane with an angle of

$$\theta_T \approx \tan\theta_T = \frac{\Delta h(r)}{\Delta r} \qquad \mathbf{2.0}$$

An exterior angle of a triangle is equal to the sum of its two interior opposite angles. Thus,

$$\beta = \theta_T - \theta_l = \frac{\Delta h(r)}{\Delta r} - \frac{h(r)}{r} \qquad \mathbf{1.0}$$

(b) The flux from of stellar radiation at distance r from the star is $E_s = \frac{L_s}{4\pi r^2}$, where L_s is the luminosity of the star. However, the irradiation flux is the normal projection of this flux onto the infinitesimal small surface A of the disk.

$$Q_+ = E_s A \sin\beta \approx \frac{L_s A\beta}{4\pi r^2} \qquad \mathbf{1.5}$$

From the Stefan-Boltzmann law, the cooling rate is give by

$$Q_- = A\sigma T_D^4. \qquad \mathbf{0.5}$$

In the thermal equilibrium

$$Q_+ = Q_- \implies T_D = \left(\frac{L_s\beta}{4\pi\sigma r^2}\right)^{\frac{1}{4}} \qquad \mathbf{1.0}$$

(c) The condition for the isothermal layer can be solved by applying

$$\begin{aligned}
\beta &= \frac{\Delta h(r)}{\Delta r} - \frac{h(r)}{r} = \frac{a(r+\Delta r)^b - ar^b}{\Delta r} - \frac{ar^b}{r} \\
&= ar^b\left(\frac{(1+\frac{\Delta r}{r})^b - 1}{\Delta r} - \frac{1}{r}\right) = ar^b\left(\frac{b\Delta r}{r\Delta r} - \frac{1}{r}\right) \\
&= a(b-1)r^{b-1} \propto r^2 \\
\implies b &= 3 \qquad \mathbf{2.0}
\end{aligned}$$

Finally, one can determine the second constant by

$$\begin{aligned}
\beta &= a(b-1)r^{b-1} = 2ar^2 \\
\therefore T_{SL}^4 &= \left(\frac{aL_s}{2\pi\sigma}\right) \\
a &= \left(\frac{2\pi\sigma T_{SL}^4}{L_s}\right) \qquad \mathbf{1.0}
\end{aligned}$$

6 Solution: Photometry of Binary stars (20 Points)

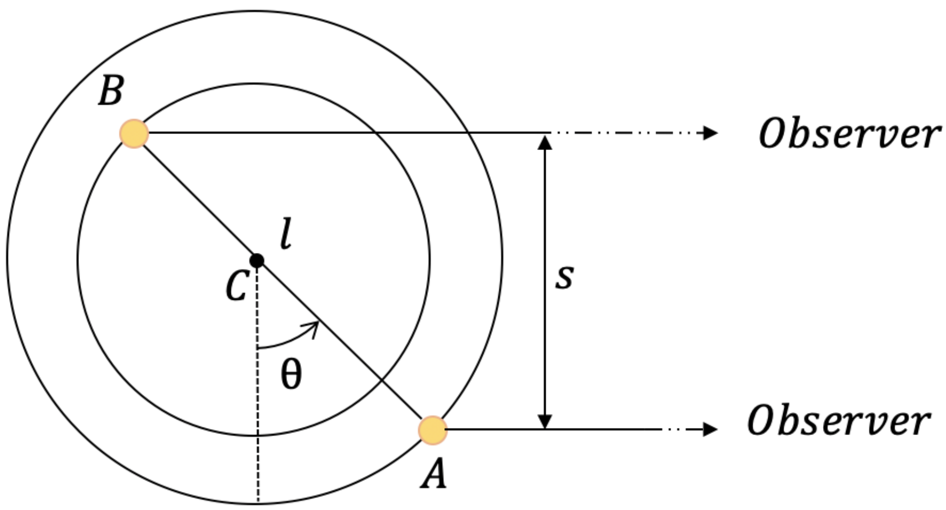
(a) From the Wien's displacement law the temperature of a black body is related to a peak wavelength of emission by the relation:

$$\lambda_{max}T = b \qquad \mathbf{1.0}$$
$$T_A = 5978K \qquad \mathbf{0.5}$$
$$T_B = 4830K \qquad \mathbf{0.5}$$

(b) Due to the diffraction minimal angular separation between two celestial objects to distinguish them:

$$\delta = 1.22\frac{\lambda}{D} = 4.44 \times 10^{-8}\,\text{rad} = 9.16\,\text{mas} \qquad \mathbf{1.0}$$

Now let us derive the time dependence of the angular separation between two stars as seen from the earth



As it is seen from the figure, distance $s = l\cos\theta$, where l is the distance between stars and θ is the angle of rotation of the stars about their center of mass after the stars are seen with maximum angular separation. The angular separation seen by an observer on the earth:

$$\gamma = \frac{s}{d} = \frac{l}{d}\cos\theta \qquad \mathbf{1.0}$$

The 38 days, during which objects are seen as a one object the system rotates by the angle $\beta = 2\pi\frac{38}{100}$.
1.0

Thus, the corresponding starting angle θ' at which stars are no longer distinguishable:

$$\theta' = \frac{\pi}{2} - \frac{\beta}{4} = \frac{\pi}{2} - \frac{19\pi}{100} = \frac{31\pi}{100}$$ **1.5**

and this is the moment, when stars stop to be distinguishable. Thus, corresponding angular separation on the sky is δ:

$$\delta = \frac{l}{d}\cos\theta'$$

From which one obtains

$$l = \frac{\delta d}{\cos\theta'} = 1.45\,\mathrm{au}$$ **1.5**

(c) From the Kepler's 3rd law:

$$T^2 = \frac{4\pi^2 l^3}{G(M_A + M_B)}$$
$$M_A + M_B = \frac{4\pi^2 l^3}{GT^2}$$
$$= 40.7 M_\odot$$ **2.0**

(d) From given data for case 1:

$$(U-V)_1 = (U-B)_1 + (B-V)_1 = 0.3$$
$$U_1 = U_{0_1} + a_U d = 6.51$$
$$V_1 = U_1 - (U-V)_1 = 6.21$$
$$m_{Bol_1} = V_1 + BC_1 = 6.31$$ **1.5**

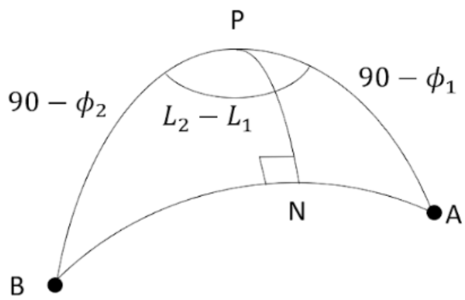
Similarly for second configuration:

$$(U-V)_2 = (U-B)_2 + (B-V)_2 = 0.37$$
$$U_2 = U_{0_2} + a_U d = 6.98$$
$$V_2 = U_2 - (U-V)_2 = 6.61$$
$$m_{Bol_2} = V_2 + BC_2 = 6.79$$ **1.0**

The difference in bolometric magnitudes:

$$
\begin{aligned}
m_{bol_2} - m_{bol_1} &= -2.5\log\left(\frac{L_2}{L_1}\right) \\
&= -2.5\log\left(\frac{\pi(R_A^2 - R_B^2)\sigma T_A^4 + \pi R_B^2\sigma T_B^4}{\pi R_A^2\sigma T_A^4 + \pi R_B^2\sigma T_B^4}\right) && \mathbf{1.5} \\
\therefore 6.79 - 6.31 &= -2.5\log\left(\frac{(R_A^2 - R_B^2)T_A^4 + R_B^2 T_B^4}{R_A^2 T_A^4 + R_B^2 T_B^4}\right) \\
&= -2.5\log\left(\frac{(R_A^2/R_B^2 - 1)T_A^4 + T_B^4}{T_A^4 R_A^2/R_B^2 + T_B^4}\right) \\
\therefore 0.48 &= -2.5\log\left[\frac{\left(\frac{R_A^2}{R_B^2} - 1\right) + \frac{T_B^4}{T_A^4}}{\frac{T_B^4}{T_A^4} + \frac{R_A^2}{R_B^2}}\right] && \mathbf{1.0} \\
\therefore 10^{\frac{-0.48}{2.5}} &= \frac{\frac{R_A^2}{R_B^2} - 1 + \frac{T_B^4}{T_A^4}}{\frac{T_B^4}{T_A^4} + \frac{R_A^2}{R_B^2}} = 10^{-0.19} = 0.65 && \mathbf{1.0} \\
\frac{R_A^2}{R_B^2} - 1 + \frac{T_B^4}{T_A^4} &= 0.65\frac{T_B^4}{T_A^4} + 0.65\frac{R_A^2}{R_B^2} \\
0.35\frac{R_A^2}{R_B^2} &= 1 - 0.35\frac{T_B^4}{T_A^4} \\
&= 1 - 0.35 \times 0.482 = 0.83 \\
\therefore \frac{R_A}{R_B} &= \sqrt{\frac{0.83}{0.35}} = 1.54 && \mathbf{1.0} \\
\frac{M_A}{M_B} &= \frac{\rho_A R_A^3}{\rho_B R_B^3} = 1.54^3 \cdot 0.7 = 2.56 && \mathbf{1.0} \\
\text{But, } M_A + M_B &= 40.7 M_\odot && \mathbf{1.0} \\
\therefore M_B &= \frac{40.7 M_\odot}{3.56} = 11.4 M_\odot && \mathbf{1.0} \\
M_B &= 29.3 M_\odot
\end{aligned}
$$

7 Solution: Georgia to Georgia (20 points)



In the diagram, N is the northernmost point of the path $\widehat{AB}$. Triangle APB, triangle APN and triangle BPN are all spherical triangles. We shall use the convention that North and West are positive, and South and East are negative. We notice,

$$\widehat{\mathrm{PN}} = -\delta_F = 30°4'$$
$$\widehat{\mathrm{PA}} = 90° - \phi_A = 48°17'$$
$$\widehat{\mathrm{PB}} = 90° - \phi_B$$
$$\sphericalangle PNA = \sphericalangle PNB = 90°$$
$$\sphericalangle BPA = \lambda_B - \lambda_A \qquad \textbf{5.0}$$
$$x = \widehat{\mathrm{AB}} = vt = 4375 \times 2\frac{17}{60} = 9990\,\mathrm{km}$$
$$\therefore x = \frac{9900}{R_\oplus} = 1.5663\,\mathrm{rad}$$
$$= 89.74° = 89°44' \qquad \textbf{2.0}$$

In $\triangle APN$, using sine rule,

$$\frac{\sin \sphericalangle A}{\sin \widehat{\mathrm{PN}}} = \frac{\sin \sphericalangle N}{\sin \widehat{\mathrm{PA}}}$$
$$\therefore \sin \sphericalangle A = \frac{\sin 90° \sin(30°4')}{\cos \phi_A} = \frac{\sin(30°4')}{\cos(41°43')} = 0.67119$$
$$\therefore \sphericalangle A = 0.7358\,\mathrm{rad} = 42.16° = 42°10' \qquad \textbf{3.0}$$

(Recognise that to have a northern point on the path, it must be the acute solution)

In $\triangle PBA$, using cosine rule,

$$\cos \widehat{\mathrm{PB}} = \cos \widehat{\mathrm{PA}} \cos \widehat{\mathrm{AB}} + \sin \widehat{\mathrm{PA}} \sin \widehat{\mathrm{AB}} \cos \sphericalangle A$$
$$\sin \phi_B = \sin \phi_A \cos x + \cos \phi_A \sin x \cos A$$
$$= \sin(41°43') \cos(89°44') + \cos(41°43') \sin(89°44') \cos(42°10')$$
$$\sin \phi_B = 0.5563$$
$$\therefore \phi_B = 0.5900\,\mathrm{rad} = 33.80° = 33°48' \qquad \textbf{4.0}$$

(Only one solution in the valid values of latitude)

Again using cosine rule,

$$\cos \widehat{\mathrm{AB}} = \cos \widehat{\mathrm{PA}} \cos \widehat{\mathrm{PB}} + \sin \widehat{\mathrm{PA}} \sin \widehat{\mathrm{PB}} \cos \sphericalangle P$$
$$\therefore \cos \sphericalangle P = \frac{\cos x - \sin \phi_A \sin \phi_B}{\cos \phi_A \cos \phi_B}$$
$$= \frac{\cos(89°44') - \sin(41°43') \sin(33°48')}{\cos(41°43') \cos(33°48')}$$
$$\cos \sphericalangle P = -0.5891$$
$$\therefore \sphericalangle P = 2.201\,\mathrm{rad} = 126.13° = 126°8'$$
$$\lambda_B = \sphericalangle P + \lambda_A = 126°8' - 41°48'$$
$$\lambda_B = 1.472\,\mathrm{rad} = 84.33° = 84°20' \qquad \textbf{5.0}$$

(The other solution of $\lambda_B - \lambda_A = -126.13°$ gives $\lambda_B = -167.93°$ which is clearly not valid)

Therefore, the coordinates of Atlanta are

$$(33°48'\mathrm{N}, 84°20'\mathrm{W}) \qquad \textbf{1.0}$$

Alternative method to find L_2
Using the spherical sine rule in triangle PBA,

$$\frac{\sin A}{\sin(90° - \phi_2)} = \frac{\sin(L_2 - L_1)}{\sin x}$$

$$\therefore \sin(L_2 - L_1) = \frac{\sin x \sin A}{\cos\phi_2}$$

$$\therefore \sin(L_2 - L_1) = \frac{\sin(89°51') \sin(42°10')}{\cos(33°43')} = 0.80689$$

$$\therefore L_2 - L_1 = 126.21° \quad (= 126°12' = 2.20\text{rad})$$

$$\therefore L_2 = 126°12' + L_1 = 126°12' + (-41°48') = 84.41° \quad (= 84°24' = 1.47\text{rad})$$

(The other solution of $L_2 - L_1 = 53.79°$ gives $L_2 = 11.99°$ which is clearly not valid)

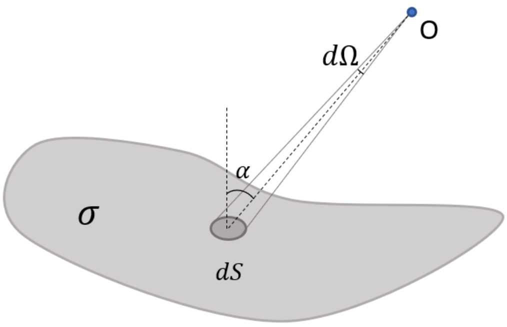
Accept all other valid solutions (such as the four parts rule or using Napier's rules) that give $\phi_2 = 33.71°$ and $L_2 = 84.41°$. Students may do both methods for calculating L_2 to confirm which solution is common to both as a quicker check than trying to do the reverse calculation with a given pair of co-ordinates.

8 Solution: Saturn's Rings (20 Points)

(a) Consider small surface element (ΔS) of the disk. The gravitational field (ΔE) generated by this surface element at at a point O located at a distance a from ΔS (see the figure below):

$$\Delta E = G\frac{\sigma \Delta S}{a^2}$$

where σ is a surface density of the mass.



where α is the angle between surface normal and the vector $\vec{r}$ and $\Delta\Omega$ is infinitesimal solid angle corresponding to the area ΔS.
For a ring, the surface density will be given by $\sigma = M/\pi(R^2 - r^2)$.
As the horizontal component will get cancelled due to symmetry considerations, we only consider the normal component:

$$\Delta E_\perp = \Delta E cos(\alpha) = \frac{G\sigma}{a^2}\Delta S cos(\alpha) = \frac{G\sigma}{a^2}\Delta S_\perp = G\sigma\Delta\Omega \qquad \textbf{5.0}$$

By Summation, we can obtain total normal field of the surface:

$$E_\perp = G\sigma\Omega \qquad \textbf{1.0}$$

here Ω is the total solid angle subtended by the disk. In our case, when we observe disk from the point of view of the test particle:

$$\Omega = 2\pi(1 - \cos\theta_{out}) - 2\pi(1 - \cos\theta_{in})$$
$$= 2\pi\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right) - 2\pi\left(1 - \frac{x}{\sqrt{x^2 + r^2}}\right)$$
$$\Omega = 2\pi x\left(\frac{1}{\sqrt{x^2 + r^2}} - \frac{1}{\sqrt{x^2 + R^2}}\right) \qquad \textbf{3.0}$$
$$\therefore F_\perp = G\sigma\Omega m$$
$$F_\perp = \frac{2GMmx}{R^2 - r^2}\left(\frac{1}{\sqrt{x^2 + r^2}} - \frac{1}{\sqrt{x^2 + R^2}}\right) \qquad \textbf{1.0}$$

(b) In small displacement approximation:

$$F_{\perp} \approx \frac{2GMmx}{R^2 - r^2}\left(\frac{1}{r} - \frac{1}{R}\right) = \left(\frac{2GMm}{Rr(R+r)}\right)x = kx \qquad \textbf{3.0}$$

when the distance between the disk and the point mass is x, the distance of the center of mass of the system from the center of the disk will be:

$$x_1 = \frac{mx}{m+M} \qquad \textbf{2.5}$$

If we choose the center of mass of the system as an origin of our coordinate system, then equation of motion of the disk will have a form:

$$Ma_1 = -kx$$
$$Ma_1 + k\frac{m+M}{m}x_1 = 0 \qquad \textbf{2.5}$$

For this simple harmonic oscillator, the period of small oscillations will be:

$$T = 2\pi\sqrt{\frac{Rr(R+r)}{2G(M+m)}} \qquad \textbf{2.0}$$

9 Solution: Solar Retrograde Motion on Mercury (20 points)

From Kepler's 3rd Law, $T^2 = a^3$ in units of Earth years and au respectively.

$$\therefore T_{\text{Mercury}} = \sqrt{(0.387)^3} = 0.240\,75\,\text{yr} = 87.934\,\text{days} \qquad \textbf{2.0}$$

We know that $\omega_{\text{orb}} = v/r$ so will vary throughout the orbit (since r varies within an ellipse) whilst ω_{rot} is constant.

$$\omega_{\text{rot}} = \frac{2\pi}{T_{\text{rot}}} = \frac{2\pi}{\frac{2}{3}\times 87.934 \times 86400} = 1.24\times 10^{-6}\,\text{rad/s} \qquad \textbf{2.0}$$

For retrograde motion, we need $\omega_{\text{orb}} \geq \omega_{\text{rot}}$.
Using the vis-viva equation for the critical value of r for when $\omega_{\text{rot}} = \omega_{\text{orb}}$,

$$\omega_{\text{orb}}^2 = \frac{v^2}{r^2} = \frac{GM_\odot}{r^2}\left(\frac{2}{r} - \frac{1}{a}\right) = \omega_{\text{rot}}^2$$
$$\frac{2GM_\odot}{r^3} - \frac{GM_\odot}{ar^2} = \omega_{\text{rot}}^2$$
$$\therefore \frac{\omega_{\text{rot}}^2}{GM_\odot}r^3 + \frac{1}{a}r - 2 = 0$$
$$(1.160\times 10^{-32})r^3 + (1.725\times 10^{-11})r - 2 = 0 \quad \text{(if } r \text{ is in meters)}$$
$$38.99r^3 + 2.584r - 2 = 0 \quad \text{(if } r \text{ is in au)} \qquad \textbf{4.0}$$

Solving the cubic equation (by any valid method) gives only one non-imaginary root
One possible way would be to use iterations for this polynomial $f(r)$, using perihelion distance as the starting guess.

r	$f(r)$
0.3073	-0.07482
0.3100	-0.03747
0.3120	-0.00967
0.3140	0.01841
0.3130	0.00433
0.3127	0.00012

$$\therefore r = 0.3127\,\text{au} = 4.684\times 10^{10}\,\text{m} \qquad \textbf{4.0}$$

From knowledge of ellipses, if E is the eccentric anomaly

$$r = a(1 - e\cos E)$$
$$\therefore E = \cos^{-1}\left[\frac{1}{e}\left(1 - \frac{r}{a}\right)\right] = \cos^{-1}\left[\frac{1}{0.206}\left(1 - \frac{0.3127}{0.387}\right)\right]$$
$$E = 0.3706\,\text{rad} = 21.23° = 21°14' \qquad \textbf{3.0}$$

Using Kepler's Equation we can find the mean anomaly, M

$$M = E - e\sin E = 0.3706 - 0.206 \times \sin 0.3706$$
$$= 0.2960\,\text{rad} = 16.96° = 16°58' \qquad \textbf{2.0}$$

This is relative to the perihelion, so symmetry demands that the total time the sun is in retrograde corresponds to

$$\Delta M = 2M$$
$$\therefore T_{\odot,retro} = T_{\text{Mercury}}\frac{\Delta M}{2\pi} = 87.934 \times \frac{2 \times 0.296}{2\pi}$$
$$T_{\odot,retro} = \boxed{8.28\,\text{days}} \qquad \textbf{3.0}$$

[Accept alternative methods making use of the true anomaly e.g. with the relation $\cos E = \frac{e+\cos\nu}{1+e\cos\nu}$.
]

10 Solution: Accretion (20 Points)

(a) Consider a particle at a distance r from the center of the compact object. In case of maximal luminosity, the gravitational force must be balanced by the radiation pressure.

$$F_G = G\frac{m_H M}{r^2} \qquad \textbf{0.5}$$
$$F_R = \frac{\sigma_e L_E}{4\pi c r^2} \qquad \textbf{0.5}$$
$$\text{But, } F_G = F_R \qquad \textbf{1.0}$$
$$\therefore L_E = \frac{4\pi G m_H M c}{\sigma_e} \qquad \textbf{1.0}$$

(b) The lower bound of its mass given by:

$$\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 m_e c^2}\right)^2$$
$$\approx 6.65 \times 10^{-29}\,\text{m}^2 \qquad \textbf{1.5}$$
$$M = \frac{\sigma_e L_\odot}{4\pi G m_H c}$$
$$= 6.04 \times 10^{25}\,\text{kg}$$
$$= 3.04 \times 10^{-5} M_\odot \qquad \textbf{2.5}$$

(c) Gravitational energy of atoms of total mass Δm:

$$\Delta E = \frac{GM\Delta m}{R}$$
$$L_{\text{acc}} = \frac{\Delta E}{\Delta t}$$
$$\therefore L_{\text{acc}} = \frac{GM}{R}\frac{\Delta m}{\Delta t} = \frac{GM}{R}\dot{M} \qquad \textbf{2.0}$$

(d) For maximum $\dot{M}$:

$$L_{\text{acc}} = L_E$$
$$\frac{GM}{R}\dot{M} = \frac{4\pi G m_H M c}{\sigma_e}$$
$$\dot{M} = \left(\frac{4\pi m_H c}{\sigma_e}\right) R \qquad \textbf{2.0}$$

(e) For given value of R_*:

$$\dot{M} = \left(\frac{4\pi m_H c}{\sigma_e}\right) R_*$$
$$\dot{M} = 1.14 \times 10^{21}\,\mathrm{kg/s}$$
$$= 3.59 \times 10^{28}\,\mathrm{kg/yr}$$
$$\dot{M} = 0.018\,\mathrm{M_\odot/yr} \qquad \mathbf{2.0}$$

(f) The total angular momentum and total mass must be conserved.

$$L = M_1 r_1^2 \Omega + M_2 r_2^2 \Omega \qquad \mathbf{1.0}$$
$$\text{Let } M_T = M_1 + M_2$$
$$\text{and } a = r_1 + r_2$$
$$r_1 = \frac{M_2 a}{M_T} \quad \text{and} \quad r_2 = \frac{M_1 a}{M_T} \qquad \mathbf{1.0}$$
$$\therefore L = \left[M_1\left(\frac{M_2 a}{M_T}\right)^2 + M_2\left(\frac{M_1 a}{M_T}\right)^2\right]\Omega$$
$$= \left(M_1 M_2^2 + M_2 M_1^2\right)\frac{a^2\Omega}{M_T^2}$$
$$\therefore L = \frac{a^2\Omega M_1 M_2}{M_T} == \frac{a^2\Omega M_1(M_T - M_1)}{M_T} \qquad \mathbf{1.0}$$
$$\therefore a = \left(\sqrt{\frac{LM_T}{\Omega}}\right) M_1^{-0.5}(M_T - M_1)^{-0.5}$$
$$a = KM_1^{-0.5}(M_T - M_1)^{-0.5} \qquad \mathbf{1.0}$$

Now let us say mass of the compact object **increases** by small fraction ΔM_1. Let the corresponding **increases** in the separation be Δa.

$$a + \Delta a = K(M_1 + \Delta M_1)^{-0.5}(M_T - M_1 - \Delta M_1)^{-0.5}$$
$$= KM_1^{-0.5}(M_T - M_1)^{-0.5}\left(1 + \frac{\Delta M_1}{M_1}\right)^{-0.5}\left(1 - \frac{\Delta M_1}{M_T - M_1}\right)^{-0.5}$$
$$a + \Delta a = a\left(1 + \frac{\Delta M_1}{M_1}\right)^{-0.5}\left(1 - \frac{\Delta M_1}{M_2}\right)^{-0.5} \qquad \mathbf{1.0}$$
$$\therefore 1 + \frac{\Delta a}{a} \approx \left(1 - \frac{\Delta M_1}{2M_1}\right)\left(1 + \frac{\Delta M_1}{2M_2}\right)$$
$$\approx 1 - \frac{\Delta M_1}{2M_1} + \frac{\Delta M_1}{2M_2}$$
$$\therefore \frac{\Delta a}{a} = \frac{\Delta M_1}{2}\left(\frac{M_1 - M_2}{M_1 M_2}\right) \qquad \mathbf{1.0}$$

Thus, Δa is positive (separation increases) when $M_1 > M_2$ and the separation decreased when $M_1 < M_2$. **1.0**

11 Solution: Dyson Sphere (50 Points)

(a) Heat absorbed must be fully emitted to maintain the thermal equilibrium

$$kL_\odot = 4\pi R^2 \epsilon\sigma T_{\mathrm{eq}}^4 \qquad \mathbf{2.0}$$
$$\therefore T_{\mathrm{eq}} = \sqrt[4]{\frac{kL_\odot}{4\pi R^2\epsilon\sigma}} \qquad \mathbf{1.0}$$

(b) In this part, we try minimize the radius at the expanse of reaching the highest operational temperature for panels. Again

$$kL_\odot = 4\pi R^2 \epsilon \sigma T_{\max}^4$$

$$R = \sqrt{\frac{kL_\odot}{4\pi\epsilon\sigma T_{\max}^4}} \qquad \mathbf{1.0}$$

$$\approx 1 \times 10^{11}\,\mathrm{m} = 0.665\,\mathrm{au} \qquad \mathbf{2.0}$$

Clearly, this distance is approximately 1.5 times smaller than Earth-Sun separation. Hence, Earth will stay **OUT**side the sphere. **1.0**

(c) Power transmitted into usable energy

$$P = \eta L_\odot$$

$$P = 7.65 \times 10^{25}\,\mathrm{W} \qquad \mathbf{2.0}$$

(d) Energy harnessed in 1 second will be enough for

$$\tau = \frac{7.65 \times 10^{25}\,\mathrm{W} \times 1\,\mathrm{s}}{17 \times 10^{12}\,\mathrm{W}} = 4.5 \times 10^{12}\,\mathrm{s} \approx 143\,000\,\mathrm{yr} \qquad \mathbf{2.0}$$

(e) New equilibrium temperature may be found as

$$kL_\odot \frac{\pi R_\oplus^2}{4\pi a_\oplus^2} = 4\pi R_\oplus^2 \sigma T_{\mathrm{new}}^4 \qquad \mathbf{2.0}$$

$$T_{\mathrm{new}} = \sqrt[4]{\frac{kL_\odot}{16\pi\sigma a_\oplus^2}}$$

$$T_{\mathrm{new}} \approx 206\,\mathrm{K} \qquad \mathbf{2.0}$$

$$\Delta T = 288 - 206 = 82\,\mathrm{K} \qquad \mathbf{1.0}$$

(f) By Kepler's Third Law

$$T^2 = \frac{4\pi^2}{GM_\odot} R^3$$

$$\therefore T = (0.665)^{1.5}\mathrm{yr}$$

$$T \approx 0.542\,\mathrm{yr} = 198\,\mathrm{days} \qquad \mathbf{3.0}$$

(g) We compare the two forces for given area A of the solar panel

$$F_{\mathrm{RP}} = \frac{I_{inc}A}{c} = \frac{L_\odot A}{4\pi R^2 c} \qquad \mathbf{2.0}$$

$$F_{\mathrm{Grav}} = \frac{GM_\odot A\rho}{R^2} \qquad \mathbf{1.0}$$

$$\alpha = \frac{F_{\mathrm{RP}}}{F_{\mathrm{Grav}}} = \frac{L_\odot}{4\pi c GM_\odot \rho}$$

$$\alpha \approx 7.65 \times 10^{-4} \qquad \mathbf{2.0}$$

At the first sight, this might seem like a negligible effect, but let us look at the change of the orbital period

$$(1-\alpha)F_{\mathrm{Grav}} = m\omega_1^2 R \qquad \mathbf{3.0}$$

The radius should be the same, because our aim is to minimize the radius.

$$\omega_1 = \sqrt{\frac{GM_\odot(1-\alpha)}{R^3}}$$

$$T_1 = 2\pi\sqrt{\frac{R^3}{GM_\odot}}(1-\alpha)^{-0.5} \qquad \mathbf{1.0}$$

$$\therefore \frac{T_1}{T} = (1-\alpha)^{-0.5} \approx 1 + 0.5\alpha$$

$$\Delta T = 0.5\alpha T$$

$$\Delta T \approx 1.8\,\mathrm{h} \qquad \mathbf{3.0}$$

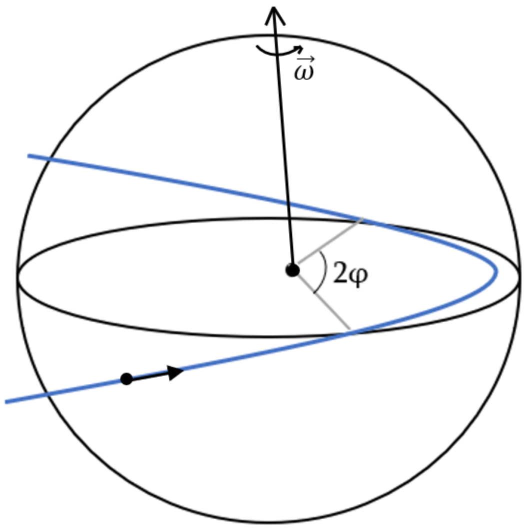


Figure 1: Transit of a asteroid through the sphere.

Therefore, **YES** this additional force has an easily detectable effect. **1.0**

(h) First, we find the angle between entrance and exit points

$$r = \frac{a}{1-\cos\theta}$$
$$R - R\cos\theta = a$$
$$\cos\theta = \frac{R-a}{R} = \frac{0.665-1}{0.665} \quad \textbf{1.0}$$
$$\cos\theta = -0.5046 \quad \textbf{1.0}$$
$$\theta_1 = 2.099\,\text{rad} = 120.2^\circ = 120^\circ 14'$$
$$\theta_2 = 4.185\,\text{rad} = 239.8^\circ = 239^\circ 46'$$
$$2\phi = 2.086\,\text{rad} = 119.5^\circ = 119^\circ 32' \quad \textbf{3.0}$$

We know the that asteroid will stay inside the sphere for about $\tau_0 = 37$ days. During this time, sphere will rotate with an angle

$$\Delta\gamma = \omega\tau_0 = \frac{2\pi\tau_0}{T}$$
$$\Delta\gamma \approx 1.17\,\text{rad} \quad \textbf{2.0}$$

Hence, if the angular distance between holes is β, the asteroid to go through safely, the following condition must be satisfied

$$2\phi = \Delta\gamma + \beta \quad \textbf{5.0}$$
$$\therefore \beta = 2\phi - \Delta\gamma$$
$$\beta \approx 0.91\,\text{rad} = 52.3^\circ \quad \textbf{1.0}$$

.

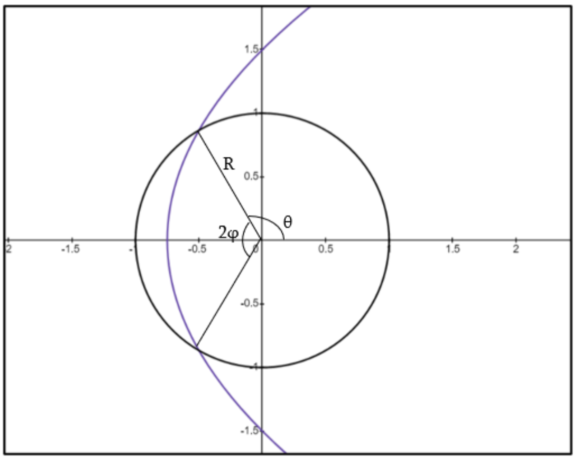


Figure 2: Transit of a asteroid in a polar coordinate system.

(i) By Wien's law, the most probable wavelength radiated by the sphere would be

$$\lambda_1 = \frac{b}{T_1}$$

$$\lambda_2 = \frac{b}{T_2} \qquad \mathbf{2.0}$$

$$\lambda_1' \approx \lambda_1 \left(1 + \frac{dH_0}{c}\right)$$

$$\lambda_2' \approx \lambda_2 \left(1 + \frac{dH_0}{c}\right) \qquad \mathbf{2.0}$$

$$\frac{b}{T_2}\left(1 + \frac{dH_0}{c}\right) < \lambda_{obs} < \frac{b}{T_1}\left(1 + \frac{dH_0}{c}\right) \qquad \mathbf{1.0}$$

12 Solution: Co-orbital satellites (50 Points)

(a) By Kepler's third law we have

$$T_i^2 = \frac{4\pi^2}{GM} r_i^3$$

$$\omega_i = \frac{2\pi}{T_i} = \sqrt{\frac{GM}{r_i^3}} \qquad \mathbf{1.5}$$

$$L_i = m_i \omega_i r_i^2$$

$$\therefore L_i = m_i \sqrt{GMr_i} \qquad \mathbf{1.5}$$

(b) When the satellites are opposing positions (i.e. $\theta = \pi$), we have

$$r_1 = R \pm \frac{x_1}{2}, \ \ r_2 = R \mp \frac{x_2}{2} \qquad \mathbf{2.0}$$

Thus, using the conservation of angular momentum

$$\begin{aligned} L_{tot} &= m_1\sqrt{GM\left(R+\frac{x_1}{2}\right)} + m_2\sqrt{GM\left(R-\frac{x_2}{2}\right)} \\ &= m_1\sqrt{GM\left(R-\frac{x_1}{2}\right)} + m_2\sqrt{GM\left(R+\frac{x_2}{2}\right)} \qquad \textbf{4.0} \\ \therefore m_1\left(1+\frac{x_1}{4R}\right) + m_2\left(1-\frac{x_2}{4R}\right) &= m_1\left(1-\frac{x_1}{4R}\right) + m_2\left(1+\frac{x_2}{4R}\right) \\ m_1\frac{x_1}{2R} &= m_2\frac{x_2}{2R} \\ \therefore \frac{m_1}{m_2} &= \frac{x_2}{x_1} \qquad \textbf{2.0} \end{aligned}$$

(c) Let the centre of mass of the ternary system be denoted by C. The total gravitational force on satellite m_2 due to the primary mass M, and the satellite m_1 may be resolved into a radial component, parallel to the line between m_2 and C and a tangential component, perpendicular to the line between m_2 and C. Only the tangential component of the force will act to change the angular momentum of the satellite m_2.
The tangential component of the force is given by

$$F_{2_\perp} = \frac{GMm_2}{r_2^2}\sin(\delta\beta) - \frac{Gm_1m_2}{d^2}\sin\beta \qquad \textbf{2.0}$$

where $d = dist(m_1, m_2)$, $\beta = \measuredangle m_1m_2C$, and $\delta\beta = \measuredangle Mm_2C$. There are two ways to simplify this expression.
The first way straightforward. By immediately exploiting the fact that m_1, $m_2 \ll M$ and hence $\delta\beta \to 0$, we can say,

$$\begin{aligned} \frac{d}{\sin\theta} &= \frac{r_1}{\sin(\beta+\delta\beta)} \\ \sin\beta &= \left(\frac{r_1}{d}\right)\sin\theta \qquad \textbf{3.0} \\ \text{Also, } \frac{\left(\frac{r_1m_1}{M+m_1}\right)}{\sin(\delta\beta)} &= \frac{r_2}{\sin(180-\theta-\delta\beta)} \\ \therefore \sin(\delta\beta) &= \left(\frac{m_1r_1}{r_2(m_1+M)}\right)\sin\theta \qquad \textbf{4.0} \end{aligned}$$

Another way is using the sine and cosine rules for triangles, we have the expressions

$$\begin{aligned} \frac{r_2}{\sin(180-\gamma)} &= \frac{\left[\left(\frac{r_1m_1}{M+m_1}\right)^2 + r_2^2 - \frac{2m_1r_1r_2}{M+m_1}\cos\theta\right]^{\frac{1}{2}}}{\sin\theta} \\ \frac{r_2\sin\theta}{\sin\gamma} &= \left[r_2^2 + \left(\frac{r_1m_1}{M+m_1}\right)^2 - \frac{2m_1r_1r_2}{M+m_1}\cos\theta\right]^{\frac{1}{2}} \\ \frac{d}{\sin\gamma} &= \frac{\frac{Mr_1}{m_1+M}}{\sin\beta} \\ \sin\beta &= \frac{Mr_1}{m_1+M}\frac{r_2}{d}\sin\theta\left[r_2^2 + \left(\frac{m_1r_1}{m_1+M}\right)^2 - \frac{2m_1r_1r_2}{m_1+M}\cos\theta\right]^{-\frac{1}{2}} \approx \frac{r_1}{d}\sin\theta \\ \sin(\delta\beta) &= \frac{m_1r_1}{m_1+M}\sin\theta\left[r_2^2 + \left(\frac{m_1r_1}{m_1+M}\right)^2 - \frac{2m_1r_1r_2}{m_1+M}\cos\theta\right]^{-\frac{1}{2}} \approx \frac{m_1r_1}{r_2(m_1+M)}\sin\theta \end{aligned}$$

In both cases, finally we obtain

$$\begin{aligned} F_{2_\perp} &= \frac{GMm_2}{r_2^2}\frac{m_1r_1}{r_2(m_1+M)}\sin\theta - \frac{Gm_1m_2}{d^2}\frac{r_1}{d}\sin\theta \\ \text{As, } M+m_1 &\approx M \\ \therefore F_{2_\perp} &= Gm_1m_2r_1\sin\theta(r_2^{-3} - d^{-3}) \qquad \textbf{3.0} \end{aligned}$$

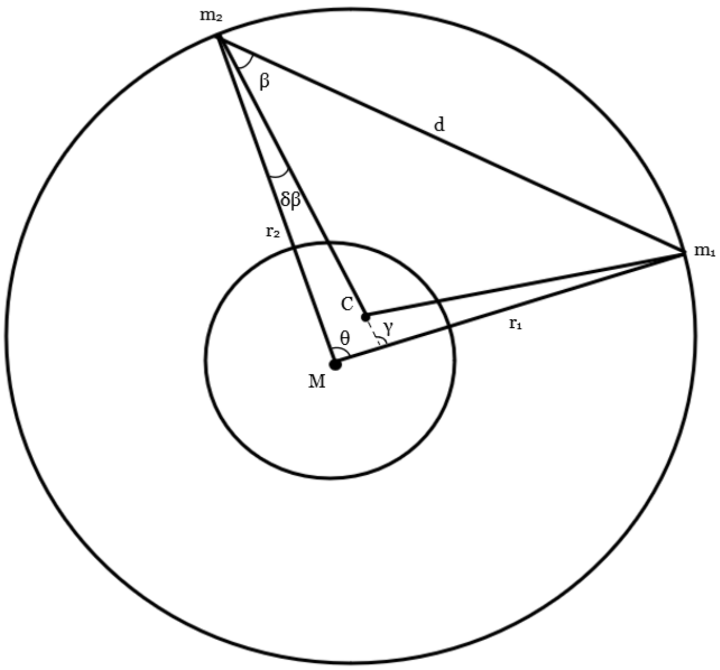


Figure 3: Configuration of the satellites and planet. Circle is centered at C(all bodies are rotating around this point).

The distance between satellites

$$d \approx 2R\sin\left(\frac{\theta}{2}\right) \qquad \textbf{1.0}$$

The tangential component of the gravitational force then becomes

$$F_{2_\perp} \approx \frac{Gm_1m_2}{R^2}\sin\theta\left(1-\frac{1}{8\sin^3\left(\frac{\theta}{2}\right)}\right)$$

$$F_{2_\perp} = \frac{Gm_1m_2}{R^2}\left(\sin\theta-\frac{\cos\left(\frac{\theta}{2}\right)}{4\sin^2\left(\frac{\theta}{2}\right)}\right) \qquad \textbf{2.0}$$

Finally, the torque generated by the gravitational force is then

$$\frac{\Delta L_2}{\Delta t} = F_{2_\perp}r \approx -\frac{Gm_1m_2}{R}\left(\frac{\cos\left(\frac{\theta}{2}\right)}{4\sin^2\left(\frac{\theta}{2}\right)}-\sin\theta\right) \qquad \textbf{3.0}$$

(d) We can see that

$$
\begin{aligned}
\Delta L_i &= m_i\sqrt{GM(r_i+\Delta r_i)} - m_i\sqrt{GMr_i} \\
&= m_i\sqrt{GMr_i}\left[\left(1+\frac{\Delta r_i}{r_i}\right)^{\frac{1}{2}} - 1\right] \\
&= m_i\sqrt{GMr_i}\left(\frac{\Delta r_i}{2r_i}\right) \\
\therefore \frac{\Delta L_i}{\Delta t} &= \frac{1}{2}m_i\sqrt{\frac{GM}{r_i}}\frac{\Delta r_i}{\Delta t} && \mathbf{2.0} \\
\frac{\Delta r_i}{\Delta t} &= \frac{2}{m_i}\sqrt{\frac{r_i}{GM}}\frac{\Delta L_i}{\Delta t} \\
\therefore \frac{\Delta r_2}{\Delta t} &\approx -\frac{2}{m_2}\sqrt{\frac{r_2}{GM}}\frac{Gm_1m_2}{R}h(\theta) \\
\frac{\Delta r_2}{\Delta t} &\approx -2m_1\sqrt{\frac{G}{MR}}h(\theta) && \mathbf{1.0}
\end{aligned}
$$

Again, since $r1 \approx r2 \approx R$, we can say by symmetry/the conservation of angular momentum

$$
\begin{aligned}
\frac{\Delta r_1}{\Delta t} &\approx 2m_2\sqrt{\frac{G}{MR}}h(\theta) && \mathbf{3.0} \\
\therefore \frac{\Delta s}{\Delta t} &= \frac{\Delta r_2}{\Delta t} - \frac{\Delta r_1}{\Delta t} \\
\frac{\Delta s}{\Delta t} &= -2\sqrt{\frac{G}{MR}}(m_1+m_2)h(\theta) && \mathbf{2.0}
\end{aligned}
$$

(e) Since θ is the angle between the two satellites, we have

$$
\begin{aligned}
\frac{\Delta\theta}{\Delta t} &= \omega_2 - \omega_1 = \sqrt{\frac{GM}{r_2^3}} - \sqrt{\frac{GM}{r_1^3}} && \mathbf{2.0} \\
&= \sqrt{\frac{GM}{R^3}}\left[\left(\frac{r_2}{R}\right)^{-\frac{3}{2}} - \left(\frac{r_1}{R}\right)^{-\frac{3}{2}}\right] \\
&= \sqrt{\frac{GM}{R^3}}\left[\left(1+\frac{r_2-R}{R}\right)^{-\frac{3}{2}} - \left(1+\frac{r_1-R}{R}\right)^{-\frac{3}{2}}\right] \\
&= \sqrt{\frac{GM}{R^3}}\left[1-\frac{3}{2}\frac{(r_2-R)}{R} - 1 + \frac{3}{2}\frac{(r_1-R)}{R}\right] \\
&\approx \sqrt{\frac{GM}{R^3}}\left[\frac{3}{2}\frac{r_1-r_2}{R}\right] \\
\frac{\Delta\theta}{\Delta t} &\approx -\frac{3}{2}\sqrt{\frac{GM}{R^3}}\frac{s}{R} && \mathbf{3.0}
\end{aligned}
$$

(f) Using the expressions in the previous two parts,

$$
\begin{aligned}
\frac{\Delta s}{\Delta\theta} &= \frac{\Delta s}{\Delta t}\cdot\frac{\Delta t}{\Delta\theta} && \mathbf{1.0} \\
&= 2\sqrt{\frac{G}{MR}}(m_1+m_2)h(\theta)\cdot\frac{2}{3}\sqrt{\frac{R^3}{GM}}\frac{R}{s} \\
&= \frac{4}{3}\frac{R^2}{M}(m_1+m_2)h(\theta)\left(\frac{1}{s}\right) \\
\therefore s\Delta s &= \frac{4R^2}{3}\frac{(m_1+m_2)}{M}h(\theta)\Delta\theta && \mathbf{1.0}
\end{aligned}
$$

(g) The minimum distance of 13 000 km corresponds to a minimum angle of

$$
\theta_{min} \approx \frac{13000}{150000} \approx 0.0868\,\mathrm{rad} = 4°58' \qquad \mathbf{2.0}
$$

Then we substitute the given values the into given expression and the result of (b) which gives

$$\frac{m_2}{m_1} \approx 3.6 \qquad \textbf{1.0}$$

and

$$m_1 + m_2 \approx 2.5 \times 10^{18}\,\text{kg} \qquad \textbf{2.0}$$

Finally

$$m_1 \approx 5.3 \times 10^{17}\,\text{kg} \quad m_2 \approx 1.9 \times 10^{18}\,\text{kg} \qquad \textbf{1.0}$$

13 Solution: Relativistic Beaming (50 Points)

(a) Energy of a photon:

$$E = hf$$

while the momentum:

$$\vec{p} = \frac{hf}{c}\vec{n} \qquad \textbf{2.0}$$

where $\vec{n}$ is an unit vector in the direction of the motion of the photon. Substituting this into energy-momentum transformation law we have:

$$\begin{aligned}
\frac{hf_L}{c} &= \gamma\left(\frac{hf_R}{c} + p_{x_R}\frac{v}{c}\right) \\
p_{x_L} &= \gamma\left(p_{x_R} + \frac{hf_R v}{c^2}\right) \\
p_{y_L} &= p_{y_R} \\
p_{z_L} &= p_{z_R}
\end{aligned} \qquad \textbf{2.0}$$

in our case:

$$\begin{aligned}
p_{x_L} &= |p_L|\cos\theta_L, & p_{y_L} &= |p_L|\sin\theta_L, & p_{z_L} &= 0 \\
p_{x_R} &= |p_R|\cos\theta_R, & p_{y_R} &= |p_R|\sin\theta_R, & p_{z_R} &= 0
\end{aligned} \qquad \textbf{3.0}$$

where

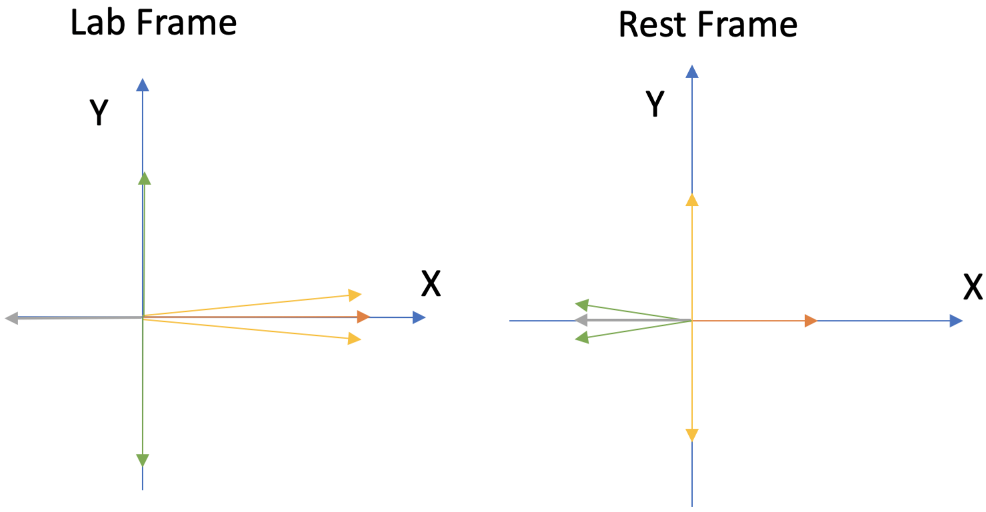
$$|p| = \frac{hf}{c}$$

Thus,

$$\begin{aligned}
f_L &= \gamma\left(f_R + \frac{v}{h}|p_R|\cos\theta_R\right) \\
&= \gamma\left(f_R + \frac{v}{h}\frac{hf_R}{c}\cos\theta_R\right) \\
\therefore f_L &= \gamma f_R\left(1 + \frac{v}{c}\cos\theta_R\right)
\end{aligned} \qquad \textbf{2.0}$$

$$\begin{aligned}
cos(\theta_L) &= \frac{p_{x_L}}{|p_L|} = \frac{cp_{x_L}}{hf_L} \\
&= \frac{c\gamma\left(p_{x_R} + \frac{hf_R v}{c^2}\right)}{h\gamma f_R\left(1 + \frac{v}{c}\cos\theta_R\right)} \\
&= \frac{c\left(\frac{hf_R}{c}\cos\theta_R + \frac{hf_R v}{c^2}\right)}{hf_R\left(1 + \frac{v}{c}\cos\theta_R\right)} \\
\therefore \cos\theta_L &= \frac{\cos\theta_R + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta_R}
\end{aligned} \qquad \textbf{2.0}$$

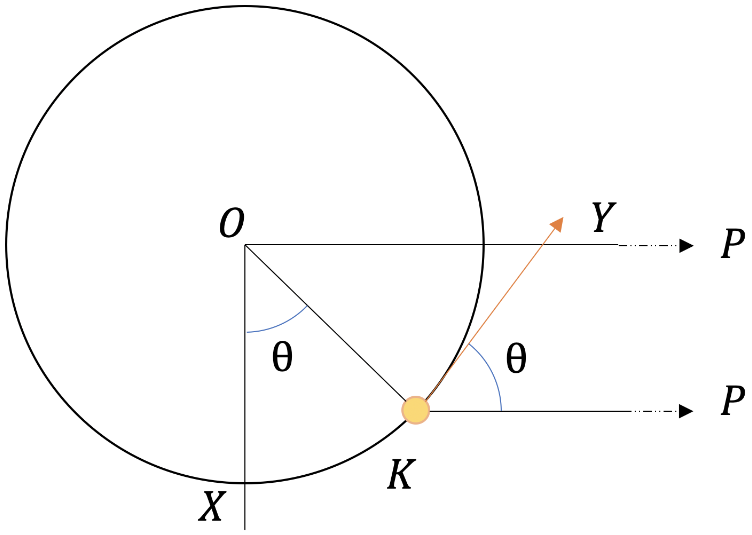
(b) Now just by plugging in the values of cosine we see that in case (i) (orange arrow) $\cos\theta_L = 1$, So the photon keeps moving in the same direction, in case (ii) (Green arrow) $\cos\theta_L = 0$, in case (iii) (Yellow arrow) $\cos\theta_L = v/c$,(iv) (Grey arrow) $\cos\theta_L = -1$.



Marking scheme,
0.5 pt for each answer in the rest frame and lab frame.
(c) From the figure we see that:

$$KP = OP - OK\sin\theta = R - r\sin\theta \qquad \mathbf{1.0}$$

where $\theta = \omega t$.



Let $t_L = T$ be the time at which photon reaches P after leaving K at a time t (in lab frame) then:

$$c(T - t) = KP = R - r\sin\theta \approx R$$
$$\therefore t = t_L - \frac{R}{c} \qquad \mathbf{1.0}$$

This photon makes angle $\theta = \omega t$ to the direction of its motion in lab frame. Now,

$$\cos\theta_L = \frac{\cos\theta_R + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta_R}$$

$$\cos\theta_L + \frac{v}{c}\cos\theta_R\cos\theta_L = \cos\theta_R + \frac{v}{c}$$

$$\cos\theta_R(1 - \frac{v}{c}\cos\theta_L) = \cos\theta_L - \frac{v}{c}$$

$$\therefore \cos\theta_R = \frac{\cos\theta_L - \frac{v}{c}}{1 - \frac{v}{c}\cos\theta_L} \qquad \textbf{2.0}$$

$$\begin{aligned} f_L &= \gamma f_R\left(1 + \frac{v}{c}\cos\theta_R\right) \\ &= \gamma f_R\left(1 + \frac{\frac{v}{c}\left(\cos\theta_L - \frac{v}{c}\right)}{1 - \frac{v}{c}\cos\theta_L}\right) \\ &= \gamma f_R\left(\frac{1 - \frac{v}{c}\cos\theta_L + \frac{v}{c}\cos\theta_L - \left(\frac{v}{c}\right)^2}{1 - \frac{v}{c}\cos\theta_L}\right) \\ &= \frac{f_R}{\gamma\left(1 - \frac{v}{c}\cos\theta_L\right)} = \frac{f_R}{\gamma\left(1 - \frac{v}{c}\cos(\omega t)\right)} \end{aligned}$$

$$\therefore f_L = \frac{f_R}{\gamma\left(1 - \frac{v}{c}\cos\left[\omega\left(t_L - \frac{R}{c}\right)\right]\right)} \qquad \textbf{3.0}$$

(d)

$$\begin{aligned} \Delta\Omega_R &= -\Delta(\cos\theta_R)\cdot\Delta\phi \\ &= [\cos\theta_R - \cos(\theta_R + \Delta\theta_R)]\cdot\Delta\phi \\ &= [\cos\theta_R - \cos\theta_R\cos(\Delta\theta_R) + \sin\theta_R\sin(\Delta\theta_R)]\cdot\Delta\phi \\ &= [\cos\theta_R - \cos\theta_R\cdot(1) + \sin\theta_R(\Delta\theta_R)]\cdot\Delta\phi \\ \Delta\Omega_R &= \sin\theta_R(\Delta\theta_R)(\Delta\phi) \\ \therefore \Delta\Omega_L &= \sin\theta_L(\Delta\theta_L)(\Delta\phi) \qquad \textbf{3.0} \\ \sin^2\theta_L &= 1 - \cos^2\theta_L \\ &= 1 - \left(\frac{\cos\theta_R + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta_R}\right)^2 \\ &= \frac{1 + 2\frac{v}{c}\cos\theta_R + \left(\frac{v}{c}\right)^2\cos^2\theta_R - \cos^2\theta_R - 2\frac{v}{c}\cos\theta_R - \left(\frac{v}{c}\right)^2}{\left(1 + \frac{v}{c}\cos\theta_R\right)^2} \\ &= \frac{\left(1 - \left(\frac{v}{c}\right)^2\right)(1 - \cos^2\theta_R)}{\left(1 + \frac{v}{c}\cos\theta_R\right)^2} = \frac{\sin^2\theta_R}{\gamma^2\left(1 + \frac{v}{c}\cos\theta_R\right)^2} \\ \sin\theta_L &= \frac{\sin\theta_R}{\gamma\left(1 + \frac{v}{c}\cos\theta_R\right)} \qquad \textbf{3.0} \end{aligned}$$

$$\text{Also, } \Delta\theta_L \approx \sin(\Delta\theta_L) = \frac{\sin(\Delta\theta_R)}{\gamma\left(1+\frac{v}{c}\cos\theta_R\right)}$$

$$\Delta\theta_L \approx \frac{\Delta\theta_R}{\gamma\left(1+\frac{v}{c}\cos\theta_R\right)} \qquad \textbf{2.0}$$

$$\frac{\Delta\Omega_L}{\Delta\Omega_R} = \frac{\sin\theta_L(\Delta\theta_L)(\Delta\phi)}{\sin\theta_R(\Delta\theta_R)(\Delta\phi)}$$

$$= \frac{\sin\theta_R(\Delta\theta_R)}{\gamma^2\left(1+\frac{v}{c}\cos\theta_R\right)^2}\cdot\frac{1}{\sin\theta_R(\Delta\theta_R)}$$

$$\therefore \Delta\Omega_L = \frac{\Delta\Omega_R}{\gamma^2\left(1+\frac{v}{c}\cos\theta_R\right)^2} \qquad \textbf{2.0}$$

(e) Let $N(\theta_L)\Delta\Omega_L\Delta T$ be the number of photons arriving in the vicinity of P within the element of solid angle $\Delta\Omega_L$ and time interval $T, T+\Delta T$ (it is important that $\Delta T \neq \Delta t$ where Δt is the emission time of the same photons (in lab frame)). These being photons emitted by K into the solid angle $\Delta\Omega$ in the time interval $t_{0R}, t_{0R}+\Delta t_{0R}$, so that:

$$N(\theta_L)\Delta\Omega_L\Delta T = N\Delta\Omega_R\Delta t_{0R} \qquad \textbf{2.0}$$

From an earlier part:

$$cT = ct + R - r\sin\theta_L = ct + R - r\sin(\omega t)$$

$$\therefore c\Delta T = c\Delta t - r[\sin(\omega t)\cos(\omega\Delta t) + \sin(\omega\Delta t)\cos(\omega t) - sin(\omega t)]$$

$$c\Delta T = c\Delta t - r\omega\cos(\omega t)\Delta t \qquad \textbf{2.0}$$

$$\frac{\Delta T}{\Delta t_{0R}} = \frac{\Delta t}{\Delta t_{0R}}\frac{\Delta T}{\Delta t} = \gamma\left(1-\frac{v}{c}\cos(\omega t)\right)$$

$$\frac{\Delta t_{0R}}{\Delta T} = \frac{1}{\gamma\left(1-\frac{v}{c}\cos(\omega t)\right)} \qquad \textbf{3.0}$$

$$\frac{N(t_L)}{N_R} = \frac{\Delta\Omega_R\Delta t_{0R}}{\Delta\Omega_L\Delta T}$$

$$= \frac{\gamma^2\left(1+\frac{v}{c}\cos\theta_R\right)^2}{\gamma\left(1-\frac{v}{c}\cos(\omega t)\right)}$$

$$= \frac{\gamma\left(1+\left(\frac{v}{c}\right)\frac{\cos\theta_L-\frac{v}{c}}{1-\frac{v}{c}\cos\theta_L}\right)^2}{\left(1-\frac{v}{c}\cos\theta_L\right)}$$

$$= \frac{\gamma}{\gamma^4\left(1-\frac{v}{c}\cos\theta_L\right)^3}$$

$$N(t_L) = \frac{N_R}{\gamma^3\left(1-\frac{v}{c}\cos\left(t_L-R/c\right)\right)^3} \qquad \textbf{3.0}$$

In terms of energy fluxes

$$F_0 = \frac{hf_R N_R}{R^2} = \frac{L}{4\pi R^2}$$

$$F(t_L) = \frac{hf_L N(t_L)}{R^2} \qquad \textbf{2.0}$$

$$\frac{F(t_L)}{F_0} = \frac{hf_L N(t_L)}{hf_R N_R}$$

$$= \frac{1}{\gamma^4 \left(1 - \frac{v}{c}\cos\left[\omega\left(t - \frac{R}{c}\right)\right]\right)^4}$$

$$F(t_L) = \frac{L}{4\pi R^2 \gamma^4 \left(1 - \frac{v}{c}\cos\left[\omega\left(t - \frac{R}{c}\right)\right]\right)^4} \qquad \textbf{3.0}$$

The radiation is strongly beamed in the direction of motion of the source so that a remote observer in or near the orbital plane of the source sees strongly pulsed radiation.
(f) The amplification (for a given v) is highest when $\cos\theta_L = 1$. **1.0**

$$F(t_L) = \frac{F_0}{\left[\gamma\left(1 - \frac{v}{c}\right)\right]^4}$$

$$A_{\max} = \frac{1}{[0.0975 \times 0.05]^4}$$

$$A_{\max} \approx 1.8 \times 10^9 \qquad \textbf{1.0}$$

$$\text{Similarly, } A_{\min} = \frac{1}{[0.0975 \times 1.95]^4}$$

$$A_{\min} \approx 770 \qquad \textbf{1.0}$$